JANUARY, 1895.

ANNALS OF MATHEMATICS.

ORMOND STONE, Editor.

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OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 9, Number 2.

ALL COMMUNICATIONS should be addressed to ORMOND STONE, University Station, Charlottesville, Va., U. S. A.

Entered at the Post Office as second-class mail matter.

LITERAL EXPRESSION FOR THE MOTION OF THE MOON'S PERIGEE.

By Dr. G. W. Hill, West Nyack, N. Y.

The earlier investigators of the Lunar Theory contented themselves with giving numerical values for this quantity. The results of Clairaut, D'Alembert, Euler, Laplace and Damoiseau are of this nature. Beyond the rudest approximation, Plana was the first to give the value in a literal form. This was nearly reproduced by Pontécoulant. The salient portion, which is a function of the ratio of the month to the year, is given by these two authors to the order of the seventh power of this ratio. Delaunay, in his treatment of the subject, (*Comptes Rendus*, t. LXXIV, p. 19) has added the two following terms, viz. those which involve the eighth and ninth powers of the mentioned ratio. The correctness of these has, however, been called in question by M. Andoyer (*Annales de la Faculté des Sciences de Toulouse*, t. VI) who has also given other values for them. They may be seen in Tisserand's *Mécanique Céleste*, t. III, p. 412. I have not been able to consult M. Andoyer's memoir, and do not know what method he used in obtaining his results. The comparison I made at the end of my memoir "On the part of the motion of the Lunar Perigee, etc. (Acta Math., Vol. VIII) of Delaunay's series with my numerical value, indicated with some probability, that the newly added terms were, one or both, too large; which corresponds with what M. Andoyer has found. In this state of matters I have thought it would not be without interest to test the validity of M. Andoyer's corrections; and I have determined to add two more terms to the series, viz. those factored by m^{10} and m^{11}.

For this purpose, I shall employ the method of my above-mentioned memoir. There the computation was carried out in the numerical fashion, here it is proposed to give algebraic developments. It is there shown that the determination of the lunar inequalities of the type of the evection and the motion of the perigee depend on, at least as far as a first approximation is concerned, the integration of the linear differential equation of the second order

$$D^2\mathrm{w} = \theta\mathrm{w}\,,$$

where w is the unknown, θ a periodic function of double the mean angular distance of the Moon from the Sun, involving only cosines, and D is an operator such that $D(a\zeta^\nu) = \nu a\zeta^\nu$, ζ being the trigonometrical exponential corresponding to the mentioned mean angular distance. The motion of the perigee depends solely on the coefficients of θ, and these can be found when

we know the coefficients of the inequalities of the type of the variation. I have given the latter in a literal form to the 9th order inclusive (Amer. Jour. Math., Vol. I, pp. 142–143).

We adopt a moving system of rectangular coordinates, the origin being at the centre of the Earth, the axis of x constantly passing through the centre of the Sun, and, in place of x and y, we use the imaginary coordinates

$$u = x + y\sqrt{-1}\,, \quad s = x - y\sqrt{-1}\,.$$

Then

$$u = \Sigma \mathrm{a}_i \zeta^{2i+1}\,, \quad s = \Sigma \mathrm{a}_i \zeta^{-2i-1}\,,$$

where, in the summation, i receives all integral values from $-\infty$ to $+\infty$, zero included, and the a_i are constants, being each equivalent to the same constant multiplied by a function of the ratio of the month to the year, which is of the $|2i|$ th order with respect to this parameter. For simplicity in writing, then, we assume that the value of a_0 is unity; consequently, as written, a_i always denotes $\frac{\mathrm{a}_i}{\mathrm{a}_0}$.

In pushing the development of Θ to the degree of approximation we desire, the values of the a_i given (Amer. Jour. Math., Vol. I, pp. 142–143) generally suffice; but it will be perceived from the approximate expression for Θ (Motion of Lunar Perigee, p. 13) that it will be necessary for the determination of c, the ratio of the anomalistic to the synodic month, to the 11th order inclusive, that we should know the term factored by m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$; it is not necessary, however, that a_1 and a_{-1} separately should be known to this degree of approximation. Hence, we now proceed to obtain this term. From the equations given (Am. Jour. Math., Vol. I, p. 137) by neglecting all terms whose order exceeds the 10th we derive

$$\begin{aligned}\mathrm{a}_1 + \mathrm{a}_{-1} = &-\frac{3(2+\mathrm{m})\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(1 + 2\,\mathrm{a}_1\mathrm{a}_{-1}) - \frac{3(1-\mathrm{m})\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(\mathrm{a}_{-1}{}^2 + 2\mathrm{a}_{-2} + 2\mathrm{a}_1\mathrm{a}_{-3}) \\ &- \frac{22-4\mathrm{m}+\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(\mathrm{a}_1\mathrm{a}_2 + \mathrm{a}_{-1}\mathrm{a}_{-2}) - 9\,\mathrm{a}_2\mathrm{a}_3\,.\end{aligned}$$

By substituting in the right member of this the values of the a_i in powers of m it is found that the term in m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$ is

$$+\frac{1605921808447}{2^{18}\,.\,3^{8}\,.\,5^{3}}\,\mathrm{m}^{10}\,.$$

We have

$$\begin{aligned}\Theta = &-\left[\frac{z}{r^3} + \mathrm{m}^2\right] + 2\left[D\log\sqrt{\frac{Du}{Ds}} + \mathrm{m}\right]^2 - [D\log\sqrt{DuDs}]^2 \\ &- D^2\,[\log\sqrt{DuDs}]\,.\end{aligned}$$

The first term of this can be developed from the formula

$$\frac{\varkappa}{r^3} + m^2 = \frac{D^2u + 2mDu + \frac{3}{2}m^2s}{u} + \frac{5}{2}m^2 = 1 + 2m + \frac{5}{2}m^2 + \Sigma \, . \, R_i \zeta^{2i},$$

where we have $R_{-i} = R_i$. The equations which determine the values of the R_i, R_0 with an error of the 12th order, R_1 with one of the 11th order, R_2 with one of the 9th order, and R_3 with one of the 7th order, are

$$R_0 + (a_1 + a_{-1}) R_1 + (a_2 + a_{-2}) R_2 = \tfrac{3}{2} m^2 a_{-1},$$

$$a_{-1}R_0 + (1 + a_{-2}) R_1 + (a_1 + a_{-3}) R_2 + a_2 R_3 = -4ma_{-1} + \tfrac{3}{2} m^2,$$

$$a_{-2}R_0 + (a_{-1} + a_{-3}) R_1 + R_2 + a_1 R_3 = 8(1 - m) a_{-2} + \tfrac{3}{2} m^2 a_1,$$

$$a_{-1}R_2 + R_3 = 24a_{-3} + \tfrac{3}{2} m^2 a_2.$$

Solving these by successive approximations in using the known literal values of the a_i, we get

$$R_0 = -\frac{9}{2^5} m^4 + 4m^5 + \frac{34}{3} m^6 + 15m^7 + \frac{2704801}{2^{13} . 3^3} m^8 + \frac{122957}{2^5 . 3^4 . 5} m^9 + \frac{1260881}{2^9 . 3^4 . 5} m^{10} - \frac{291394307}{2^7 . 3^6 . 5^3} m^{11},$$

$$R_1 = \frac{3}{2} m^2 + \frac{19}{2^2} m^3 + \frac{20}{3} m^4 + \frac{43}{3^2} m^5 + \frac{18709}{2^9 . 3^3} m^6 + \frac{759413}{2^{10} . 3^4 . 5} m^7 + \frac{6675059}{2^8 . 3^5 . 5^2} m^8 - \frac{41991161}{2^7 . 3^6 . 5^3} m^9 - \frac{4528083484913}{2^{17} . 3^7 . 5^4} m^{10},$$

$$R_2 = \frac{33}{2^4} m^4 + \frac{2937}{2^6 . 5} m^5 + \frac{23051}{2^4 . 3 . 5^2} m^6 + \frac{97051}{2^5 . 5^3} m^7 + \frac{334413271}{2^{10} . 3^3 . 5^4} m^8,$$

$$R_3 = \frac{1393}{2^9} m^6.$$

We next attend to the coefficients of $\frac{D^2u}{Du} = \Sigma \, . \, U_i \zeta^{2i}$. The formulas given for these (Motion of Lunar Perigee, p. 12) in general suffice; it is necessary, however, to push the development of U_1 and U_{-1} farther, so that terms of the 10th order may be included. Thus their equivalents read

$$U_1 = 2\,[h_1 - h_{-1} h_2 + h_1^2 h_{-1} + 2h_1^3 h_{-1}^2 - h_1^3 h_{-2} - 3h_1 h_{-1}^2 h_2 + 2h_1 h_2 h_{-2} + h_{-1}^2 h_3 - h_{-2} h_3],$$

$$U_{-1} = -2\,[h_{-1} - h_1 h_{-2} + h_{-1}^2 h_1 + 2h_{-1}^3 h_1^2 - h_{-1}^3 h_2 - 3h_{-1} h_1^2 h_{-2} + 2h_{-1} h_{-2} h_2 + h_1^2 h_{-3} - h_2 h_{-3}].$$

By substituting in these equations the values of the a_i in powers of m, and making the assumption that the coefficient of m^{10} in a_1 is 0, which cannot lead us into error within the limits we set to the approximation, and putting

$$\tfrac{1}{2}(U_i + U_{-i}) = A_i\,, \quad \tfrac{1}{2}(U_i - U_{-i}) = B_i\,,$$

we get

$$A_1 = -\frac{5}{2^3}\mathrm{m}^2 - \frac{1}{2\,.\,3}\mathrm{m}^3 + \frac{5}{3^2}\mathrm{m}^4 + \frac{43}{2^2\,.\,3^3}\mathrm{m}^5 - \frac{318575}{2^{11}\,.\,3^4}\mathrm{m}^6 - \frac{2297593}{2^8\,.\,3^5\,.\,5}\mathrm{m}^7$$

$$-\frac{9225887}{2^5\,.\,3^6\,.\,5^2}\mathrm{m}^8 - \frac{3471983789}{2^9\,.\,3^7\,.\,5^3}\mathrm{m}^9 - \frac{12903700736069}{2^{19}\,.\,3^8\,.\,5^4}\mathrm{m}^{10}\,,$$

$$B_1 = \frac{7}{2^2}\mathrm{m}^2 + \frac{19}{2\,.\,3}\mathrm{m}^3 + \frac{53}{2\,.\,3^2}\mathrm{m}^4 + \frac{155}{2^2\,.\,3^3}\mathrm{m}^5 - \frac{12941}{2^{10}\,.\,3^4}\mathrm{m}^6 - \frac{904921}{2^9\,.\,3^5\,.\,5}\mathrm{m}^7$$

$$-\frac{35308207}{2^9\,.\,3^6\,.\,5^2}\mathrm{m}^8 - \frac{2190838913}{2^{10}\,.\,3^7\,.\,5^3}\mathrm{m}^9 + \frac{29589760583167}{2^{18}\,.\,3^8\,.\,5^4}\mathrm{m}^{10}\,,$$

$$A_2 = \frac{265}{2^7}\mathrm{m}^4 + \frac{1067}{2^5\,.\,5}\mathrm{m}^5 + \frac{38261}{2^4\,.\,3^2\,.\,5^2}\mathrm{m}^6 + \frac{755591}{2^6\,.\,3^2\,.\,5^3}\mathrm{m}^7 + \frac{405840581}{2^{10}\,.\,3^4\,.\,5^4}\mathrm{m}^8\,,$$

$$B_2 = -\frac{3}{2^2}\mathrm{m}^4 - \frac{403}{2^4\,.\,3\,.\,5}\mathrm{m}^5 - \frac{3773}{2^3\,.\,3^2\,.\,5^2}\mathrm{m}^6 - \frac{246139}{2^5\,.\,3^3\,.\,5^3}\mathrm{m}^7 - \frac{1077852389}{2^{12}\,.\,3^4\,.\,5^4}\mathrm{m}^8\,,$$

$$A_3 = -\frac{1677}{2^{11}}\mathrm{m}^6\,,$$

$$B_3 = \frac{2431}{2^{10}}\mathrm{m}^6\,.$$

The coefficients of the function θ have the following equivalents:

$$\theta_0 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2 + 4(A_1^2 + A_2^2) + 2(B_1^2 + B_2^2) - R_0\,,$$

$$\theta_1 = 4(1 + \mathrm{m})A_1 - 2B_1 + 4(A_1A_2 + A_2A_3) + 2(B_1B_2 + B_2B_3) - R_1\,,$$

$$\theta_2 = 4(1 + \mathrm{m})A_2 - 4B_2 + 2(A_1^2 + 2A_1A_3) - B_1^2 + 2B_1B_3 - R_2\,,$$

$$\theta_3 = 4A_3 - 6B_3 + 4A_1A_2 - 2B_1B_2 - R_3\,.$$

When the expressions in powers of m for the quantities A, B, and R are substituted in the preceding equations, the results are

$$\theta_0 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2 + \frac{255}{2^5}\mathrm{m}^4 + 19\mathrm{m}^5 + \frac{80}{3}\mathrm{m}^6 + \frac{533}{2\,.\,3^2}\mathrm{m}^7 + \frac{11230225}{2^{13}\,.\,3^3}\mathrm{m}^8$$

$$+ \frac{1576037}{2^7\,.\,3^4}\mathrm{m}^9 + \frac{49359583}{2^9\,.\,3^5}\mathrm{m}^{10} + \frac{720508007}{2^8\,.\,3^6\,.\,5}\mathrm{m}^{11}\,,$$

$$\theta_1 = -\frac{15}{2}\,\mathrm{m}^2 - \frac{57}{2^2}\,\mathrm{m}^3 - 11\mathrm{m}^4 - \frac{23}{2\,.\,3}\,\mathrm{m}^5 - \frac{68803}{2^9\,.\,3^2}\,\mathrm{m}^6 - \frac{1792417}{2^{10}\,.\,3^3}\,\mathrm{m}^7$$

$$-\frac{7172183}{2^7\,.\,3^4\,.\,5}\,\mathrm{m}^8 - \frac{596404499}{2^9\,.\,3^5\,.\,5^2}\,\mathrm{m}^9 - \frac{2641291011773}{2^{17}\,.\,3^6\,.\,5^3}\,\mathrm{m}^{10},$$

$$\theta_2 = \frac{111}{2^4}\,\mathrm{m}^4 + \frac{1397}{2^6}\,\mathrm{m}^5 + \frac{8807}{2^4\,.\,3\,.\,5}\,\mathrm{m}^6 + \frac{319003}{2^5\,.\,3^2\,.\,5^2}\,\mathrm{m}^7 + \frac{252382507}{2^{10}\,.\,3^3\,.\,5^3}\,\mathrm{m}^8,$$

$$\theta_3 = -\frac{11669}{2^9}\,\mathrm{m}^6.$$

We employ now the system of equations given (Motion of the Lunar Perigee, p. 14) and for brevity of notation put $[i]$ for $(c + i)^2 - \theta_0$. The equations, written to the requisite degree of approximation, are

$$\begin{array}{rrrrrrl}
[-3]\,b_{-3} & -\theta_1 b_{-2} & -\theta_2 b_{-1} & -\theta_3 b_0 & & & = 0\,, \\
-\theta_1 b_{-3} & +[-2]\,b_{-2} & -\theta_1 b_{-1} & -\theta_2 b_0 & -\theta_3 b_1 & & = 0\,, \\
-\theta_2 b_{-3} & -\theta_1 b_{-2} & +[-1]\,b_{-1} & -\theta_1 b_0 & -\theta_2 b_1 & -\theta_3 b_2 & = 0\,, \\
-\theta_3 b_{-3} & -\theta_2 b_{-2} & -\theta_1 b_{-1} & +[0]\,b_0 & -\theta_1 b_1 & -\theta_2 b_2 & = 0\,, \\
 & -\theta_3 b_{-2} & -\theta_2 b_{-1} & -\theta_1 b_0 & +[1]\,b_1 & -\theta_1 b_2 & = 0\,, \\
 & & -\theta_3 b_{-1} & -\theta_2 b_0 & -\theta_1 b_1 & +[2]\,b_2 & = 0\,.
\end{array}$$

That the relative degree of importance of the terms of these equations may be perceived, it may be pointed out that the diagonal line of coefficients $[-3]$, $[-2], \ldots, [1], [2]$ are all of the zero order of magnitude except $[-1]$ and $[0]$, the first of which is of the first order, and the second of the third order; but the latter we need not concern ourselves about; and θ_i is of the $2i$th order. From this it follows that, if we write the quantities b in the order b_{-1}, b_1, b_{-2}, b_2, b_{-3}, leaving out b_0, which is an arbitrary quantity, the first is of the first order, and every succeeding one an order higher, so that b_{-3} is of the fifth order. In order to have the equation determining c, it is necessary to eliminate the 5 mentioned b's from the group of equations. The readiest method of accomplishing this is to proceed by successive approximations using formulas of recursion. To attain the desired degree of accuracy, three approximations are necessary, each of which will give three terms in powers of m in the value of each b involved. When the values of the five b's have been obtained and substituted in the middle equation, after the rejection of the useless factor

b_0, we have the following equation serving for the determination of c :

$$
\begin{aligned}
[0] &- \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \left[\frac{1}{[-1]^2[-2]} + \frac{1}{[1]^2[2]}\right]\theta_1^4 \\
&- 2\left[\frac{1}{[-1][1]} + \frac{1}{[-1][-2]} + \frac{1}{[1][2]}\right]\theta_1^2\theta_2 \\
&- \left[\frac{1}{[-2]} + \frac{1}{[2]}\right]\theta_2^2 - \frac{1}{[-1]^2}\left[\frac{1}{[-1][-2]^2} + \frac{1}{[-2]^2[-3]}\right]\theta_1^6 \\
&- 2\left[\frac{1}{[-1]^2[-2]^2} + \frac{1}{[-1]^2[1][-2]} + \frac{1}{[-1][1]^2[2]}\right. \\
&\qquad \left. + \frac{1}{[-1]^2[-2][-3]} + \frac{1}{[-1][-2]^2[-3]}\right]\theta_1^4\theta_2 \\
&- \left[\frac{1}{[-1][-2]^2} + \frac{1}{[-1]^2[1]} + \frac{1}{[-1][1]^2} + \frac{1}{[-1]^2[-3]}\right. \\
&\qquad \left. + \frac{2}{[-1][1][-2]} + \frac{2}{[-1][1][2]} + \frac{2}{[-1][-2][-3]}\right]\theta_1^2\theta_2^2 \\
&- 2\left[\frac{1}{[-1][-2][-3]} + \frac{1}{[-1][1][-2]} + \frac{1}{[-1][1][2]}\right]\theta_1^3\theta_3 \\
&- 2\left[\frac{1}{[-1][2]} + \frac{1}{[-1][-3]}\right]\theta_1\theta_2\theta_3 = 0\,.
\end{aligned}
$$

From this all terms unnecessary to the desired degree of approximation have been rejected.

It appears desirable to give some details as to the treatment of the foregoing equation. First we form the various products of the θ involved ; each is limited to the terms needed for the degree of approximation wished.

$$
\begin{aligned}
\theta_1^2 = \frac{225}{2^2}\,m^4 + \frac{855}{2^2}\,m^5 + \frac{5889}{2^4}\,m^6 + 371m^7 + \frac{697679}{2^9 . 3}\,m^8 + \frac{853817}{2^6 . 3^2}\,m^9 \\
+ \frac{235899233}{2^{11} . 3^3}\,m^{10} + \frac{1733519201}{2^9 . 3^4 . 5}\,m^{11} + \frac{199791349395549}{2^{18} . 3^5 . 5^2}\,m^{12},
\end{aligned}
$$

$$
\begin{aligned}
\theta_1^4 = \frac{50625}{2^4}\,m^8 + \frac{192375}{2^3}\,m^9 + \frac{2787075}{2^5}\,m^{10} + \frac{6370695}{2^5}\,m^{11} \\
+ \frac{353456169}{2^{10}}\,m^{12} + \frac{649258747}{2^{10}}\,m^{13},
\end{aligned}
$$

$$\theta_1^6 = \frac{11390625}{2^6} m^{12} + \frac{129853125}{2^6} m^{13} + \frac{2868159375}{2^8} m^{14},$$

$$\theta_1^2\theta_2 = \frac{24975}{2^6} m^8 + \frac{693945}{2^8} m^9 + \frac{1188267}{2^7} m^{10} + \frac{21446525}{2^{10}} m^{11}$$

$$+ \frac{4710472379}{2^{13}.3.5} m^{12},$$

$$\theta_2^2 = \frac{12321}{2^8} m^8 + \frac{155067}{2^9} m^9 + \frac{20185533}{2^{12}.5} m^{10} + \frac{85123117}{2^9.3.5^2} m^{11},$$

$$\theta_1^4\theta_2 = \frac{5619375}{2^8} m^{12} + \frac{241552125}{2^{10}} m^{13}, \quad \theta_1^2\theta_2^2 = \frac{2772225}{2^{10}} m^{12} + \frac{55958985}{2^{11}} m^{13},$$

$$\theta_1^3\theta_3 = \frac{39382875}{2^{12}} m^{12}, \qquad \theta_1\theta_2\theta_3 = \frac{19428885}{2^{14}} m^{12}.$$

In the next place, by neglecting quantities of the 7th order, the equation may be written

$$[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \frac{\theta_1^4}{128m^2} = 0,$$

and, if we put

$$\theta'_1 = \theta_1\left[1 - \frac{3\,\theta_1^2}{64m}\right],$$

it can be given the form

$$[0]^2 + 2(\theta_0 - 1)[0] + \theta'^2_1 = 0,$$

whence is derived

$$c^2 = 1 + \sqrt{(\theta_0 - 1)^2 - \theta'^2_1}.$$

By substituting the previously given developments of θ_0 and θ_1 we get

$$c^2 = 1 + 2m - \frac{1}{2} m^2 - \frac{225}{2^4} m^3 - \frac{3135}{2^6} m^4 - \frac{139973}{2^{10}} m^5 - \frac{4611319}{2^{12}.3} m^6,$$

$$c = 1 + m - \frac{3}{2^2} m^2 - \frac{201}{2^5} m^3 - \frac{2367}{2^7} m^4 - \frac{111749}{2^{11}} m^5 - \frac{4095991}{2^{13}.3} m^6.$$

These equations are correct to the last power of m set down.

This value of c may be substituted in all the terms but the two first of the equation which determines it; and the latter is thereby reduced to a manageable form. The values of the reciprocals of the quantities denoted by the

symbols [— 1], [1], [— 2], [2], [— 3], developed to the needed degree of approximation, are

$$\frac{1}{[-1]} = -\frac{1}{2^2}m^{-1} - \frac{3}{2^4} - \frac{213}{2^8}m - \frac{2259}{2^{10}}m^2 - \frac{70973}{2^{13}}m^3 - \frac{3501259}{2^{15}.3}m^4,$$

$$\frac{1}{[1]} = \frac{1}{2^3} - \frac{1}{2^4}m + \frac{5}{2^6}m^2 + \frac{563}{2^{10}}m^3 + \frac{6119}{2^{12}}m^4,$$

$$\frac{1}{[-2]} = \frac{1}{2^3} + \frac{1}{2^3}m + \frac{1}{2^5}m^2 - \frac{643}{2^{10}}m^3 - \frac{10807}{2^{12}}m^4 - \frac{532047}{2^{16}}m^5,$$

$$\frac{1}{[2]} = \frac{1}{2^3.3} - \frac{1}{2^3.3^2}m + \frac{13}{2^5.3^3}m^2 + \frac{8557}{2^{10}.3^4}m^3,$$

$$\frac{1}{[-3]} = \frac{1}{2^3.3} + \frac{1}{2^4.3}m.$$

The substitution being made in the eight terms of the left-hand member of the equation, the result follows, in which, for facility of verification, we write each fraction separately and in the order in which it arises from each of the eight terms.

$$-\frac{50625}{2^{11}}m^6 + \left[-\frac{1022625}{2^{12}} + \frac{24975}{2^9}\right]m^7$$

$$+\left[-\frac{90037575}{2^{16}} + \frac{49095}{2^7} - \frac{4107}{2^9}\right]m^8$$

$$+\left[-\frac{1462100355}{2^{18}} + \frac{54632079}{2^{15}} - \frac{57165}{2^{10}} + \frac{11390625}{2^{18}}\right]m^9$$

$$+\left[-\frac{165044625741}{2^{23}} + \frac{722443913}{2^{17}} - \frac{8198151}{2^{13}.5} + \frac{705459375}{2^{20}} - \frac{13111875}{2^{17}}\right.$$
$$\left. - \frac{924075}{2^{15}}\right]m^{10}$$

$$+\left[-\frac{287970294069}{2^{22}} + \frac{87619247043}{2^{20}.5} - \frac{13794117581}{2^{17}.3^2.5^2} + \frac{92222296875}{2^{24}}\right.$$
$$\left. - \frac{702232875}{2^{19}} - \frac{34533765}{2^{17}} + \frac{65638125}{2^{19}} + \frac{6476295}{2^{17}}\right]m^{11}.$$

By summing the fractions the equation may be written

$$[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 = \frac{50625}{2^{11}}\mathrm{m}^6 + \frac{822825}{2^{12}}\mathrm{m}^7 + \frac{65426631}{2^{16}}\mathrm{m}^8$$
$$+ \frac{514143669}{2^{17}}\mathrm{m}^9 + \frac{579596224169}{2^{23} \,.\, 5}\mathrm{m}^{10} + \frac{182494574380633}{2^{24} \,.\, 3^2 \,.\, 5^2}\mathrm{m}^{11}.$$

It will be more suitable for solution if both members are multiplied by

$$-\tfrac{1}{8}[-1][1] = 4\mathrm{m} - \mathrm{m}^2 - \frac{225}{2^4}\mathrm{m}^3 - \frac{2625}{2^6}\mathrm{m}^4 - \frac{120517}{2^{10}}\mathrm{m}^5 - \frac{4587389}{2^{12} \,.\, 3}\mathrm{m}^6.$$

The right member then becomes

$$\frac{50625}{2^9}\mathrm{m}^7 + \frac{1595025}{2^{11}}\mathrm{m}^8 + \frac{112880037}{2^{15}}\mathrm{m}^9 + \frac{1422559539}{2^{17}}\mathrm{m}^{10}$$
$$+ \frac{137176160137}{2^{20} \,.\, 5}\mathrm{m}^{11} + \frac{477331474933 93}{2^{22} \,.\, 3^2 \,.\, 5^2}\mathrm{m}^{12}.$$

Calling this K, we have

$$\mathrm{c}^2 = 1 - \tfrac{1}{8}\theta_1^2 + \sqrt{(\theta_0 - 1 + \tfrac{1}{8}\theta_1^2)^2 - \theta_1^2 + K + \tfrac{1}{8}(\mathrm{c}^2 - \theta_0)^3}\,.$$

From the preceding expressions for c and θ_0,

$$\tfrac{1}{2}(\mathrm{c}^2 - \theta_0)^3 = -\frac{225}{2^5}\mathrm{m}^3 - \frac{3645}{2^7}\mathrm{m}^4 - \frac{159429}{2^{11}}\mathrm{m}^5 - \frac{1646333}{2^{13}}\mathrm{m}^6;$$

whence

$$\tfrac{1}{8}(\mathrm{c}^2 - \theta_0) = -\frac{11390625}{2^{15}}\mathrm{m}^9 - \frac{553584375}{2^{17}}\mathrm{m}^{10} - \frac{60085546875}{2^{21}}\mathrm{m}^{11}$$
$$- \frac{1228257320625}{2^{23}}\mathrm{m}^{12}.$$

The substitutions being made in the foregoing value of c^2 and the square root extracted, we get

$$\mathrm{c} = 1 + \mathrm{m} - \frac{3}{2^2}\mathrm{m}^2 - \frac{201}{2^5}\mathrm{m}^3 - \frac{2367}{2^7}\mathrm{m}^4 - \frac{111749}{2^{11}}\mathrm{m}^5 - \frac{4095991}{2^{13} \,.\, 3}\mathrm{m}^6$$
$$- \frac{332532037}{2^{16} \,.\, 3^2}\mathrm{m}^7 - \frac{15106211789}{2^{18} \,.\, 3^3}\mathrm{m}^8 - \frac{5975332916861}{2^{23} \,.\, 3^4}\mathrm{m}^9$$
$$- \frac{1547804933375567}{2^{25} \,.\, 3^5 \,.\, 5}\mathrm{m}^{10} - \frac{818293211836767367}{2^{28} \,.\, 3^6 \,.\, 5^2}\mathrm{m}^{11}.$$

By means of the equation

$$\frac{1}{n}\frac{d\omega}{dt} = 1 - \frac{\mathrm{c}}{1 + \mathrm{m}},$$

we derive from the last result the ratio of the motion of the perigee to the mean motion in longitude, viz.:

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\mathrm{m}^2 + \frac{177}{2^5}\mathrm{m}^3 + \frac{1659}{2^7}\mathrm{m}^4 + \frac{85205}{2^{11}}\mathrm{m}^5 + \frac{3073531}{2^{13}\,.\,3}\mathrm{m}^6$$

$$+ \frac{258767293}{2^{16}\,.\,3^2}\mathrm{m}^7 + \frac{12001004273}{2^{18}\,.\,3^3}\mathrm{m}^8 + \frac{4823236506653}{2^{23}\,.\,3^4}\mathrm{m}^9$$

$$+ \frac{1258410742976387}{2^{25}\,.\,3^5\,.\,5}\mathrm{m}^{10} + \frac{6672839226796009\,27}{2^{28}\,.\,3^6\,.\,5^2}\mathrm{m}^{11}\,.$$

In this we make the substitution $\mathrm{m} = \frac{m}{1 - m}$, in which m is the parameter usually employed, and prolong the resulting series only to the 9th power of m. We obtain

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}m^2 + \frac{225}{2^5}m^3 + \frac{4071}{2^7}\mathrm{m}^4 + \frac{265493}{2^{11}}m^5 + \frac{12822631}{2^{13}\,.\,3}m^6$$

$$+ \frac{1273925965}{2^{16}\,.\,3^2}m^7 + \frac{66702631253}{2^{18}\,.\,3^3}m^8 + \frac{29726828924189}{2^{23}\,.\,3^4}m^9\,.$$

The two terms ending this series are identical with M. Andoyer's and there can be no doubt as to their correctness. I do not push this series to the terms involving m^{10} and m^{11}, as I think the former in terms of the parameter m is to be preferied.

The series which has just been obtained is unsatisfactory on account of its slow convergence. It would be of great utility to transform it in such a manner that the convergence should be sensibly augmented. Here it seems no course is open but to experiment. Confining our attention to parameters of the form $\frac{\mathrm{m}}{1 - \alpha\mathrm{m}}$, we may seek the value of α which brings about the greatest improvement in convergence. It is plain that the adoption of a small value for this quantity would not sensibly change the series in this respect, but as α is augmented we shall reach a value where one of the numerical coefficients vanishes; if the latter belong to a high power of m, the adjacent coefficients will be small. This is true on the assumption that the series tends to become a geometrical progression. In the present case it appears that the coefficient of m^4 is the first to vanish with augmenting α. Desiring therefore that all the coefficients may still be positive after the transformation, I adopt a value of α which is less than the value which makes the mentioned coefficient vanish. The new parameter adopted is

$$\mathbf{m} = \frac{\mathrm{m}}{1 - \frac{3}{4}\mathrm{m}} = \frac{m}{1 - \frac{7}{4}m}\,.$$

By making the denominator of a, 4, we secure the advantage that the denominators of the coefficients of the series are not augmented. With this parameter then, we have the following series:

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\mathbf{m}^2 + \frac{141}{2^5}\mathbf{m}^3 + \frac{57}{2^5}\mathbf{m}^4 + \frac{41213}{2^{11}}\mathbf{m}^5 + \frac{243353}{2^{12}\,.\,3}\mathbf{m}^6 + \frac{84226279}{2^{16}\,.\,3^2}\mathbf{m}^7$$
$$+ \frac{1317113479}{2^{17}\,.\,3^3}\mathbf{m}^8 + \frac{1125417061277}{2^{23}\,.\,3^4}\mathbf{m}^9 + \frac{115179069708721}{2^{24}\,.\,3^5\,.\,5}\mathbf{m}^{10}$$
$$+ \frac{1065454233308527477}{2^{28}\,.\,3^6\,.\,5^2}\mathbf{m}^{11}\,.$$

The coefficients here diminish more rapidly than in the series proceeding according to powers of m.

Correspondent to the values of n and n' employed in my previous memoirs on the lunar theory we have $\mathbf{m} = 0.0860678013$. Substituting this in the series just given, we obtain the following result, exhibited term by term; it has been assumed that the series to start from the term involving $\mathbf{m}^{11}$ may be regarded as a geometrical progression having the ratio $\frac{1}{3}$,

$$\begin{aligned}\frac{1}{n}\frac{d\omega}{dt} = {} & 0.0055557498 + 0.0028092554 + 0.0000977435 + 0.0000950403 \\ & + 0.0000080501 + 0.0000049959 + 0.0000011207 + 0.0000004292 \\ & + 0.0000001260 + 0.0000000418 + \overset{\text{[remainder]}}{0.0000000209} \\ = {} & 0.0085725736\,.\end{aligned}$$

The value deduced, without resorting to any expansion in powers of a parameter, is 0.0085725730. The difference of 6 units may be attributed to the uncertainty in the estimation of the remainder or to accumulated error in forming the sum of the terms of the series.

OCT. 18, 1894.

NOTE ON GREGORY'S DISCUSSION OF THE CONDITIONS FOR AN UMBILICUS.

By Mr. Angelo Hall, Cambridge, Mass.

Gregory's treatment of the conditions for an umbilicus in Arts. 280, 292, 295, 296* would gain in simplicity by a rearrangement of the material. It will be seen that the author obtains in Art. 280 the quadratic equation for the greatest and least values of the radius of curvature at a given point on the surface, and thence derives in Art. 292 the general conditions for an umbilicus, and hence the special conditions to be satisfied at points where any one of the three partial derivatives U, V, W vanishes. But in Art. 295† it is stated that "when any of the quantities U, V, or W is equal to zero, the transformations of Art. (280) . . . is [sic] impossible, or rather nugatory"!! Again, Art. 296 says: "In the preceding Article it has been remarked that, although in certain cases the investigation of Art. 280 involves indeterminate operations, yet the equation (18) there obtained will be universally true, and that it must be essentially the same as if it had been found by a process involving no nugatory expressions. We shall, however, now give an investigation which is at all times free from nugatory operations, in order that every light may be thrown upon this important equation." Then follows a proof of equation (18) which if substituted for that given in Art. 280 would render unnecessary the latter half of each of the Arts. 292, 295. For since the quadratic in question (now derived by a method free from any indeterminateness) becomes when $U = 0$, $V \gtrless 0$, $W \gtrless 0$

$$\left\{\frac{P}{\rho} - u\right\}\left\{\frac{P}{\rho} - \frac{V^2 w + W^2 v - 2\,V W u'}{V^2 + W^2}\right\} = \frac{(V v' - W w')^2}{V^2 + W^2}\,;\ddagger$$

and since the necessary and sufficient conditions that an equation of the form $(z - a)(z - b) = c^2$ may have equal roots are $c = 0$, and $a = b$, hence the conditions for an umbilicus in this case are

$$V v' - W w' = 0\,, \quad u(V^2 + W^2) = V^2 w + W^2 v - 2\,V W u'\,.$$

Similarly may be treated other cases, such as $V = 0$, $U \gtrless 0$, $W \gtrless 0$; or $U = V = 0$, $W \gtrless 0$, etc.

When U, V, W are each not zero we may proceed according to the methods of Arts. 280, 292 (first half), and obtain formula (26), p. 264.

* A Treatise on the Application of Analysis to Solid Geometry by Gregory and Walton (Second Edition, 1852).

† Misprinted "225."

‡ See Gregory, pp. 268-9.

CONCERNING THE DEFINITION BY A SYSTEM OF FUNCTIONAL PROPERTIES OF THE FUNCTION $f(z) = \frac{\sin \pi z}{\pi}$.

By Prof. E. Hastings Moore, Chicago, Ill.

Introduction.

The definition of a function by a system (or, preferably, by various equivalent systems) of characteristic functional properties is of fundamental importance in the theory of that function, whether that theory be considered in itself or with respect to its application in the investigation of other functions or classes of functions.

I have not found in the literature consulted any such functional definition of the function $f(z) = \frac{\sin \pi z}{\pi}$, except that connected with its differential equation.

In the first part of this paper a functional definition is established for the function in question. In the second part this definition is used (for which purpose, indeed, this particular definition was hit upon) to effect a determination of the external exponential factor in the expression of the function as a Weierstrassian infinite product of primary factors.

Part I.

Definition of the function $f(z) = \frac{\sin \pi z}{\pi}$ *by a system of characteristic functional properties.*

Theorem.* *There exists one and only one function* $f(z)$, $f(z) = \frac{\sin \pi z}{\pi}$, *which possesses the following functional properties :*

($\mathbf{A}_1$) $f(z)$ *is a (transcendental) integral function of the complex variable* z.

($\mathbf{A}_2$) $f(z)$ *has as its complete system of zeros* $z = m = 0, \pm 1, \pm 2, \ldots$, *the multiplicity of each zero being unity.*

$$(\mathbf{A}_3) \qquad \lim_{z \doteq 0} \frac{f(z)}{z} = +1\,, \quad \text{or} \quad \left[\frac{d}{dz} f(z)\right]_{z=0} = +1\,.$$

*This theorem was embodied in a paper, bearing the same title as this paper, read August 15, 1894, in Brooklyn before the American Mathematical Society at its first summer meeting. In that proof (**C**) and two other properties (**D**, **E**), direct consequences of (**A**, **B**, **C**),

$$(\mathbf{D}) \qquad f(\tfrac{1}{2} + z) = f(\tfrac{1}{2} - z)\,,$$

$$(\mathbf{E}) \qquad f(z+1) = -f(z)\,,$$

were used from the beginning. The question raised by Professor Morley, of Haverford College, whether the properties (**A**, **B**, **C**) were all necessary, led to the formulation of the text which brings the essential elements of that proof into sharper relief.

$$(\mathbf{B}) \qquad f(2z)f(\tfrac{1}{2}) = 2f(z)f(z+\tfrac{1}{2})\,.$$

$$(\mathbf{C}) \qquad f(-z) = -f(z)\,.$$

It will appear that in this system of properties the property (**C**) may be replaced by

$$(\mathbf{C}') \qquad \left[\frac{d}{dz}\log\frac{f(z)}{z}\right]_{z=0} = 0\,,$$

a property in itself far less sweeping than (**C**).

That *one* function, the function

$$f(z) = \frac{\sin \pi z}{\pi} = \frac{e^{\pi i z} - e^{-\pi i z}}{2\pi i}\,,$$

possesses the properties (**A**,* **B**, **C**) or (**A**, **B**, **C′**) will be at once granted.

Before showing that *only one* function has the properties, the following lemma will be proved :—

Lemma. *The most general function $h(z)$ with the properties* (**A**, **B**) is

$$h(z) = e^{az}g(z)\,,$$

where $g(z)$ is any particular function with those properties, and where a is an arbitrary constant.

From the general theory† of integral functions it appears at once that we may set

$$h(z) = e^{l(z)}g(z)\,, \tag{1_1}$$

where $l(z)$ is any integral function of z, as an expression for the most general function $h(z)$ with the properties ($\mathbf{A}_1$, $\mathbf{A}_2$); but $h(z)$ is to satisfy ($\mathbf{A}_3$, **B**) also, whence $l(z)$ must satisfy

$$e^{l(0)} = 1\,, \tag{2_1}$$

$$e^{l(2z)+l(\frac{1}{2})} = e^{l(z)+l(z+\frac{1}{2})}\,. \tag{3_1}$$

As an allowable determination of the constant in $l(z)$ we set

$$l(0) = 0\,. \tag{$2_1'$}$$

From (3_1) we have

$$l(2z) + l(\tfrac{1}{2}) = l(z) + l(z+\tfrac{1}{2}) + \mu_z 2\pi i\,, \tag{$3_1'$}$$

where μ_z is an integer, dependent conceivably upon the particular value of z, but not really so dependent, since the other terms of ($3_1'$) are continuous func-

*(**A**) will always mean ($\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$).

† See the introduction to Part II.

tions of z. Hence we replace μ_z by μ, a constant integer, and determine its value, $\mu = 0$, by substituting in $(3_1')$ $z = 0$; thus,

$$l(2z) + l(\tfrac{1}{2}) = l(z) + l(z + \tfrac{1}{2}). \tag{$3_1''$}$$

$(3_1'')$ is an identity in z. We substitute $z + \frac{1}{2}$ for z and have another identity

$$l(2z + 1) + l(\tfrac{1}{2}) = l(z + \tfrac{1}{2}) + l(z + 1), \tag{$3_1'''$}$$

and by subtraction,

$$l(2z + 1) - l(2z) = l(z + 1) - l(z). \tag{4}$$

Write, for a moment,

$$l(z + 1) - l(z) = k(z). \tag{5}$$

$k(z)$ is an integral function of z with the identity

$$k(2z) = k(z) \tag{$4'$}$$

or

$$k(z) = k\left[\frac{z}{2}\right] = k\left[\frac{z}{2^2}\right] = \ldots = k\left[\frac{z}{2^n}\right]; \quad (n = 1, 2, 3, \ldots) \tag{$4''$}$$

whence, since $k(z)$ is a function continuous (in particular) at $z = 0$, $k(z)$ is a constant a,

$$k(z) = a = k(0); \tag{$4'''$}$$

whence, from (5) and $(2_1')$,

$$l(z + 1) - l(z) = a = l(1); \; l(0) = 0. \tag{6_1}$$

We set

$$l(z) = az + m(z), \tag{7_1}$$

so that $m(z)$ is an integral function of z for which

$$m(0) = 0, \quad m(1) = 0, \tag{2_2}$$

$$m(z + 1) - m(z) = 0, \tag{6_2}$$

$$m(2z) + m(\tfrac{1}{2}) = m(z) + m(z + \tfrac{1}{2}). \tag{3_2}$$

Introducing

$$Z(z) = e^{2\pi i z}, \; 2\pi i z = \log Z, \tag{8}$$

with the initial correspondence

$$Z = 1, \quad z = 0, \tag{9}$$

we set, thus changing the independent variable,

$$m(z) = m_*(Z), \tag{10_1}$$

and have in $m_*(Z)$ an analytic function of Z which in view of (6_2) is single-

valued, and which has singularities, if at all, only at $Z=0$, $Z=\infty$ (which correspond to $z=\infty$), and for which from $(2_2, 3_2)$

$$m_*(1)=0\,, \tag{2_3}$$

$$m_*(Z^2)+m_*(-1)=m_*(Z)+m_*(-Z)\,. \tag{3_3}$$

We may express $m_*(Z)$ as a Laurent series unconditionally convergent for every Z, except perhaps for $Z=0, \infty$,

$$m_*(Z)=\sum_{-\infty}^{+\infty}{}_i\, c_i Z^i,$$

for which the identity

$$\sum_{-\infty}^{+\infty}{}_i\, c_i Z^{2i}+\sum_{-\infty}^{+\infty}{}_i\,(-1)^i c_i=\sum_{-\infty}^{+\infty}{}_i\, c_i Z^i+\sum_{-\infty}^{+\infty}{}_i\,(-1)^i c_i Z^i \tag{3_4}$$

holds. The corresponding coefficients on left and right must be equal. The coefficients of odd powers of Z vanish on left and right. By comparison of the coefficients of Z^{2j} ($j \gtrless 0$) on left and right we have

$$2c_{2j}=c_j\,. \qquad (j \text{ is any integer except } 0) \tag{12_1}$$

This recursion formula (12_1) leads to the following determination of the c's with even suffixes $[\gtrless 0]$ in terms of the c's with odd suffixes,

$$c_{2^n\nu}=\frac{1}{2^n}\,c_\nu\,. \qquad \left(\begin{array}{l}n \text{ is any zero or positive integer,}\\ \nu \text{ is any odd integer.}\end{array}\right) \tag{12_2}$$

At first sight we seem to be almost as far as ever from the definitive determination of $m_*(Z)$, since the identity (3_4) furnishes no information about the coefficients c_ν (ν odd); but fortunately the *convergence* of the Laurent series comes to our assistance, showing that *every coefficient* c_ν (ν *odd*) *vanishes*, and hence (12_2) *every coefficient* c_i ($i \gtrless 0$) *vanishes*, and hence, since $m_*(1)=0$ (2_3), $m_*(Z)$ *vanishes identically*. For, if any particular c_ν (ν odd) were to fail to vanish, consider the partial series obtained by selecting from (11) the infinitude of terms with $i=2^n\nu$, $n=0, 1, 2, \ldots$, which by (12_2) we may write

$$c_\nu \sum_0^\infty{}_n\, \frac{1}{2^n}\, Z^{2^n\nu}\,. \tag{13}$$

This power-series should converge for the whole plane, except for $Z=\infty$ (ν positive) or for $Z=0$ (ν negative), and hence the same should be true of its term-by-term derivative series, and of that series multiplied by Z, viz. of the series

$$c_\nu\, \nu \sum_0^\infty{}_n\, Z^{2^n\nu}\,; \tag{14}$$

but this series does *not* so converge, it diverges for example for $Z = +1$. Whence the supposition that (any particular) $c_\nu \gtrless 0$ (ν odd) is recognized as untenable.

Thus we have

$$m(z) = m_*(Z) = 0\,, \tag{10_2}$$

$$l(z) = az\,, \tag{7_2}$$

$$h(z) = e^{az} g(z)\,. \tag{1_2}$$

The last equality (1_2) completes the proof of the LEMMA; clearly a is an *arbitrary* constant.

If now $g(z)$ is any particular function with the properties (**A**, **B**, **C**), the most general function $h(z)$ with those properties turns out to be $g(z)$ itself, (which is the THEOREM), since the condition (**C**) gives on the a

$$e^{-az} = e^{az}\,; \tag{15}$$

whence, by a line of argument like that used above with respect to the exponential equation (3_1), we have in fact

$$a = 0\,, \quad e^{az} = 1\,. \tag{16}$$

It is obvious that (**C**), which in the system (**A**, **B**, **C**) is needed and used only to effect the determination $a = 0$, may be replaced by (**C′**), so that the systems (**A**, **B**, **C**), (**A**, **B**, **C′**) are equivalent.

PART II.

Application: A new determination of the external exponential factor in the expression of the function $f(z) = \frac{\sin \pi z}{\pi}$ *as a Weierstrassian infinite product.*

From a knowledge of the complete system of zeros, multiplicity included, of an integral function $F(z)$ of the complex variable Z, Weierstrass* has shown how to construct in the form of an infinite product of properly determined primary factors an integral function of z with precisely the same system of zeros; the original function $F(z)$ is then the product of Weierstrass's infinite

* Weierstrass: Zur Theorie der eindeutigen analytischen Functionen (Abhandlungen der Königl. Akademie der Wissenchaften zu Berlin vom Jahre 1876). Reprinted in Abhandlungen aus der Functionenlehre, pp. 1–52 (1886).

See also: Forsyth, Theory of Functions of a Complex Variable, chapter V (1893). Harkness and Morley, A Treatise on the Theory of Functions, pp. 186–193 (1893).

product by an integral function which has no zero in the finite z-plane, and which may therefore be expressed in the form

$$e^{l(z)},$$

where $l(z)$ is an integral function of z.

In the application of Weierstrass's method to the factorisation of any particular function this "external exponential factor" demands separate determination. Picard* by use of Cauchy's theorem has effected a determination of the external exponential factor for an extensive system of functions $F(z)$, under which the function $s(z) = \frac{\sin \pi z}{\pi}$ is included. Other determinations for this particular function have been published. The one immediately to be given depends upon the Theorem of Part I, and is, so far as I know, new.

We have the function

$$s(z) = \frac{\sin \pi z}{\pi} = \frac{e^{\pi i z} - e^{-\pi i z}}{2\pi i}, \tag{1}$$

with the properties (**A**, **B**, **C**) of Part I. ($\mathbf{A}_1$, $\mathbf{A}_2$) lead us to the Weierstrassian product

$$p(z) = z\, \Pi'_m \left[1 - \frac{z}{m}\right] e^{\frac{z}{m}}, \quad (m = \pm 1, \pm 2, \ldots) \tag{2}$$

and the equation

$$s(z) = e^{l(z)} p(z). \tag{3}$$

The determination of the factor $e^{l(z)}$, $e^{l(z)} = +1$, will be effected by the direct identification by the Theorem of Part I of $s(z)$ and $p(z)$. $p(z)$ has the properties ($\mathbf{A}_1$, $\mathbf{A}_2$) to begin with, and evidently also ($\mathbf{A}_3$, **C**). It remains then merely to show that $p(z)$ has the property (**B**).

The index m shall take all integral values from $-\infty$ to $+\infty$, $m = 0$ excepted,† the indices n, n' all positive integral values, and the index ν all positive odd integral values. With this understanding, by combining into one factor the two factors of $p(z)$ with $m = +n, -n$, we have at once

$$p(z) = z\, \Pi'_m \left[1 - \frac{z}{m}\right] e^{\frac{z}{m}} = z\, \Pi_n \left[1 - \frac{z^2}{n^2}\right]; \tag{4}$$

whence

$$p(2z) = 2z\, \Pi_n \left[1 - \frac{4z^2}{n^2}\right] = 2z\, \Pi_n \left[1 - \frac{z^2}{n^2}\right] \Pi_\nu \left[1 - \frac{4z^2}{\nu^2}\right], \tag{5}$$

where we have distributed the factors of the first infinite product into the two

* Picard: Traité d'Analyse, vol. II, pp. 164–7, 1893.

† The exception is denoted after Weierstrass by the ′ in the symbol Π'.

factors according to the evenness or oddness of the n's, $n = 2n'$, $n = 2n' - 1 = \nu$, and then in the first factor replaced the index n' by the equivalent n.

$$p(\tfrac{1}{2}) = \tfrac{1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}, \tag{6}$$

$$p(z+\tfrac{1}{2}) = \frac{2z+1}{2}\, \Pi_n \left[1 - \frac{(2z+1)^2}{4n^2}\right] \tag{7}$$

$$= \frac{2z+1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}\, \Pi_n \left[\overline{1 - \frac{2z}{2n-1}} \cdot \overline{1 + \frac{2z}{2n+1}}\right] \tag{7'}$$

$$= \tfrac{1}{2}\, \Pi_n \frac{(2n-1)(2n+1)}{(2n)(2n)}.\, \Pi_\nu \left[1 - \frac{4z^2}{\nu^2}\right], \tag{7''}$$

where to pass from (7′) to (7″) we have made use of the identity

$$1 + 2z = \Pi_n \frac{1 + \dfrac{2z}{2n-1}}{1 + \dfrac{2z}{2n+1}}, \tag{8}$$

and have associated the corresponding factors of the second infinite product (7′) and the infinite product (8). From (4, 5, 6, 7″) we have for $p(z)$ the property (**B**)

$$p(2z)\, p(\tfrac{1}{2}) = 2p(z)\, p(z+\tfrac{1}{2}). \tag{9}$$

UNIVERSITY OF CHICAGO, *September* 22, 1894.

SOLUTIONS OF EXERCISES.

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SOLVE the cubic $9x^3 + 9x = 2$.

SOLUTION.

By Cardan's formula

$$\alpha = \tfrac{1}{3}(\sqrt[3]{9} - \sqrt[3]{3}), \quad \beta = \tfrac{1}{3}(\omega\sqrt[3]{9} - \omega^2\sqrt[3]{3}), \quad \gamma = \tfrac{1}{3}(\omega^2\sqrt[3]{9} - \omega\sqrt[3]{3}),$$

in which ω, ω^2 are the imaginary cube roots of unity.

[*Jesse Pawling, Jr.*]

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INTEGRATE the differential

$$\frac{\sin\theta\cos\theta\,d\theta}{\sin^4\theta + \cos^4\theta}.$$

SOLUTION.

Put the expression in terms of 2θ and let $\cos 2\theta = x$, then

$$\int\frac{\sin\theta\cos\theta\,d\theta}{\sin^4\theta + \cos^4\theta} = -\tfrac{1}{2}\int\frac{dx}{1 + x^2} = \tfrac{1}{2}\cot^{-1}\cos 2\theta.$$

[*Jesse Pawling, Jr.*]

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SHOW that the equation

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

represents a surface of revolution and find its axis. [*Geo. R. Dean.*]

SOLUTION II.

The equation may be written

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz) = a^3,$$

or

$$(x + y + z)[3(x^2 + y^2 + z^2) - (x + y + z)^2] = 2a^3,$$

or

$$(x + y + z)^3 - 3(x^2 + y^2 + z^2)(x + y + z) = -2a^3.$$

Solving for $(x + y + z)$ we have

$$(x + y + z) = \varphi(x^2 + y^2 + z^2),$$

the general equation of a surface of revolution whose axis passes through the origin and has the equation $x = y = z$. [*Geo. R. Dean.*]

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On the axis of x take any point x_m, and equidistant from it on either side select points a and b. Draw verticals through a and b, also on the circle on ab as diameter. With a and b as centers and radius equal to a side of the inscribed square draw arcs cutting the verticals at a and b in a' and b'. The straight lines $a'x_m$ and $b'x_m$ cut the circle in points whose abscissæ are x_1 and x_2, respectively.

Show that if any cubic of the form

$$y = x^3 + \alpha x^2 + \beta x + \gamma$$

be described passing through any two points x_1y_1 and x_2y_2 on the verticals at a and b respectively, then the area included between the curve, the verticals at a and b, and the x-axis, is constant and equal to

$$\tfrac{1}{2}(y_1 + y_2)(b - a).$$ [*W. H. Echols.*]

SOLUTION.

After Gauss, the area of the curve

$$y = x^3 + ax^2 + bx + c,$$

taken between the limits $x = a$, $x = b$, is

$$\int_a^b \frac{1}{\beta - \alpha} \begin{vmatrix} 0 & 1 & x \\ y_\beta & 1 & \beta \\ y_\alpha & 1 & \alpha \end{vmatrix} dx,$$

wherein, after Jacobi, α and β are the roots of

$$\left[\frac{d}{dx}\right]^2 (x-a)^2 (x-b)^2 = 0,$$

or respectively are

$$\tfrac{1}{2}(a + b) \mp \frac{1}{2\sqrt{3}}(b - a).*$$

Very evidently these values of α and β satisfy the geometrical construction indicated for locating the ordinates y_1 and y_2.

The above integral effected, regard being had for the values of α and β, we obtain very readily for the area desired

$$\tfrac{1}{2}(b - a)(y_1 + y_2).$$ [*W. H. Echols.*]

369

Solve the equation

$$\frac{d^2y}{dx^2} - y = e^{\frac{1}{2}x^2}.$$ [*Geo. R. Dean.*]

* See Boole's Calculus of Finite Differences, 2d Ed., p. 52.

SOLUTION.

Writing the equation in the form $(D^2 - 1)\,y = e^{\frac{1}{2}x^2}$, we obtain

$$y = \frac{1}{D^2 - 1}\, e^{\frac{1}{2}x^2} = \frac{1}{2}\cdot\frac{1}{D-1}\, e^{\frac{1}{2}x^2} - \frac{1}{2}\cdot\frac{1}{D+1}\, e^{\frac{1}{2}x^2}.$$

Substituting z for $\frac{1}{2}x^2$, we get

$$\frac{1}{D_x - 1}\, e^{\frac{1}{2}x^2} = \frac{1}{D_z - \frac{1}{\sqrt{2z}}}\cdot\frac{e^z}{\sqrt{2z}} = e^{\sqrt{2z}}\int e^{z-\sqrt{2z}}\,\frac{dz}{\sqrt{2z}} + c_1 e^{\sqrt{2z}},$$

$$\frac{1}{D_x + 1}\, e^{\frac{1}{2}x^2} = \frac{1}{D_z + \frac{1}{\sqrt{2z}}}\cdot\frac{e^z}{\sqrt{2z}} = e^{-\sqrt{2z}}\int e^{z+\sqrt{2z}}\,\frac{dz}{\sqrt{2z}} + c_2 e^{-\sqrt{2z}}.$$

Therefore,

$$y = c\left[e^x\int e^{\frac{1}{2}x^2 - x}\,dx - e^{-x}\int e^{\frac{1}{2}x^2 + x}\,dx\right] + c_1 e^x + c_2 e^{-x}.$$

[*Geo. R. Dean.*]

371

Prove that the ratio of

$$\begin{vmatrix} \sigma_1(2u_1), & \sigma_1(2u_2), & \sigma_1(2u_3) \\ \sigma_2(2u_1), & \sigma_2(2u_2), & \sigma_2(2u_3) \\ \sigma_3(2u_1), & \sigma_3(2u_2), & \sigma_3(2u_3) \end{vmatrix}$$

to

$$\sigma(u_2 + u_3)\,\sigma(u_3 + u_1)\,\sigma(u_1 + u_2)\,\sigma(u_2 - u_3)\,\sigma(u_3 - u_1)\,\sigma(u_1 - u_2)$$

is independent of the arguments u_λ; and that its value is

$$4\,(e_2 - e_3)\,(e_3 - e_1)\,(e_1 - e_2)\,;$$

the notation being that of Weierstrass. [*Frank Morley.*]

SOLUTION.

We know that $(\sigma_\lambda u / \sigma u)^2$ is an elliptic function of u (Harkness and Morley's Theory of Functions, p. 309, formula 35), and that $\sigma_\lambda(2u)$ is a quadratic form in $\sigma_\lambda{}^2 u$, $\sigma_\mu{}^2 u$, $\sigma_\nu{}^2 u$ (p. 316). Hence $\sigma_\lambda(2u)/\sigma^4 u$ is an elliptic function with periods $2\omega_\lambda$; and the determinant is therefore made an elliptic function of each argument by dividing it by $(\sigma u_1\ \sigma u_2\ \sigma u_3)^4$. But the same process renders the denominator of the ratio elliptic (see p. 307, formula 30). Hence the ratio is an elliptic function of each argument.

Regarded as a function of u_λ, the ratio can be ∞ only when $u_\lambda = \pm\, u_\mu$ or $\pm\, u_\nu$; for these are the zeros of the denominator. But any one of these con-

ditions makes two columns of the determinant identical. Thus, observing that these zeros are only of the first order, we see that the ratio is nowhere infinite. Hence our elliptic function of u_λ has no poles, and therefore (p. 288) it is independent of u_λ. Therefore it is a constant.

To determine this constant let $u_\lambda = \omega_\lambda$ where $\lambda = 1, 2, 3$. The determinant is (p. 312, formula 43)

$$\begin{vmatrix} -1, & 1, & 1 \\ 1, & -1, & 1 \\ 1, & 1, & -1 \end{vmatrix} \exp 2\Sigma\eta_\lambda\omega_\lambda ,$$

the periods being such that $\Sigma\omega_\lambda = 0$, $\Sigma\eta_\lambda = 0$.

The denominator is (p. 312, formula 44)

$$-(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)(\sigma\omega_1\, \sigma\omega_2\, \sigma\omega_3)^4 .$$

But (p. 311, formula 41)

$$\exp 2\Sigma\eta_\lambda\omega_\lambda = -[(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)]^2(\sigma\omega_1\, \sigma\omega_2\, \sigma\omega_3)^4 .$$

Hence the constant ratio is

$$4(e_2 - e_3)(e_3 - e_1)(e_1 - e_2) .$$ [*Frank Morley.*]

372

When the bilinear invariant of two binary n-ics is zero, we say that the n-ics are *apolar*. When also the n-ics coincide we say that either is self-apolar. And we may apply the same adjectives to the sets of n points (or n-*ads*) which represent the zeros of the n-ics. Any odd set of points is, we know, self-apolar (Salmon's Higher Algebra, § 153). Prove that an even set is self-apolar when the first polar of any point of the set, with regard to the rest, is self-apolar. [*Frank Morley.*]

SOLUTION.

Writing the equation which determines the n-ad, when $n = 2h$, in the shape

$$s_0x^{2h} - s_1x^{2h-1} + s_2x^{2h-2} - \ldots + s_{2h} = 0 , \quad (1)$$

the condition of self-apolarity is

$$(2h)!\, s_0s_{2h} - 1!\,(2h-1)!\, s_1s_{2h-1} + \ldots + (-)^h s_h^2\, h!^2/2 = 0 . \quad (2)$$

Take the selected point as origin, so that $s_{2h} = 0$; the first polar of the origin with regard to all the points is

$$s_1x^{2h-1} - 2s_2x^{2h-2} + \ldots + (2h-1)\, s_{2h-1}x = 0 ;$$

and when we remove the factor x, this gives the first polar of the origin as to the remaining $2h-1$ points. Hence, if these $2h-2$ points be given by

$$\sigma_0 x^{2h-2} - \sigma_1 x^{2h-1} + \ldots + \sigma_{2h-2} = 0\,,$$

we have

$$s_1\sigma_1 = 2s_2\sigma_0\,, \quad s_1\sigma_2 = 3s_3\sigma_0\,, \quad \ldots, \quad s_1\sigma_{2h-2} = (2h-1)\,s_{2h-1}\,\sigma_0\,;$$

therefore (2), when expressed in terms of σ, is

$$(2h-2)!\,\sigma_0\,\sigma_{2h-2} - (2h-3)!\,\sigma_1\,\sigma_{2h-3} + \ldots + (-)^{h-1}\sigma_{h-1}{}^2\,(h-1)!^2/2 = 0\,,$$

a relation which differs from (2) only in the change from h to $h-1$. Hence the theorem. A system of two points is self-apolar when the points coincide; a system of 4 points which is self-apolar is also "equi-anharmonic."

[*Frank Morley.*]

373

PROVE that in an ellipse the area of the triangle formed by the tangents at the points whose eccentric angles are α, β, γ, respectively, is

$$A = ab \tan\tfrac{1}{2}(\alpha-\beta)\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha)\,.$$

[*F. P. Matz.*]

This exercise is given by Salmon (Conic Sections, 6th ed., p. 220).

374

SHOW without resorting to the Differential Calculus that the two parabolas whose equations are $y^2 = ax$ and $x^2 = by$ intersect at an angle

$$\theta = \tan^{-1}\left[3a^{\frac{1}{3}}b^{\frac{1}{3}}/2(a^{\frac{2}{3}} + b^{\frac{2}{3}})\right]. \qquad \text{[}\textit{F. P. Matz.}\text{]}$$

SOLUTION.

At the point of intersection

$$x = a^{\frac{1}{3}}b^{\frac{2}{3}}\,, \quad y = a^{\frac{2}{3}}b^{\frac{1}{3}}\,.$$

Since the subtangent to the parabola is double the abscissa, the inclinations of the tangents at this point to the x-axis are

$$\tan\alpha = \frac{b^{\frac{1}{3}}}{2a^{\frac{1}{3}}}\,, \qquad \tan\beta = \frac{2b^{\frac{1}{3}}}{a^{\frac{1}{3}}}\,;$$

$$\therefore\ \tan\beta - \tan\alpha = \frac{3b^{\frac{1}{3}}}{2a^{\frac{1}{3}}}\,, \qquad 1 + \tan\beta\tan\alpha = 1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\,;$$

and therefore

$$\tan\theta = 3a^{\frac{1}{3}}b^{\frac{1}{3}}/\left[2(a^{\frac{2}{3}} + b^{\frac{2}{3}})\right]. \qquad \text{[}\textit{P. H. Philbrick.}\text{]}$$

Solved also by H. Y. Benedict, Geo. R. Dean, William Hoover, G. B. M. Zerr, W. O. Whitescarver, and Jesse Pawling, Jr.

375

If the hodograph of a curve described under constant acceleration be a parabola in which the radius-vectors are drawn from the focus, the intrinsic equation of the curve described is

$$s = c \int \sec^5 \tfrac{1}{2}\varphi \, d\varphi \, . \qquad \text{[F. P. Matz.]}$$

SOLUTION.

The equation of the hodograph may be written $r = k \sec^2 \frac{1}{2}\varphi$, where k is a constant. But the radius vector in the hodograph equals the velocity in orbit, or $\dfrac{ds}{dt} = k \sec^2 \frac{1}{2}\varphi$. The velocity in the hodograph

$$= \frac{d\varphi}{dt}\sqrt{r^2 + \left[\frac{dr}{d\varphi}\right]^2} = k \sec^3 \tfrac{1}{2}\varphi \, \frac{d\varphi}{dt}$$

$=$ the acceleration in the orbit $= f$, a constant. Eliminating dt and writing c for $\dfrac{k^2}{f}$, there results $ds = c \sec^5 \frac{1}{2}\varphi \, d\varphi$. [*H. Y. Benedict.*]

Solved also by Geo. R. Dean, William Hoover, and G. B. M. Zerr.

376

At each end of a horizontal base-line $2a$ the angle of elevation of a monument is θ, and at the middle point of this base-line the angle of elevation is φ. Prove that the elevation of the monument is

$$H = \frac{a \sin \varphi \sin \theta}{\sqrt{[\sin(\varphi + \theta)\sin(\varphi - \theta)]}} \, . \qquad \text{[F. P. Matz.]}$$

SOLUTION.

The projection of the line from the end of the base to the top of the monument is $H \cot \theta$, and the projection of the line from the middle of the base is $H \cot \varphi$. The latter is evidently perpendicular to the base.

$$\therefore \; H^2 \cot^2 \theta - H^2 \cot^2 \varphi = a^2 \, , \tag{1}$$

$$H^2 = \frac{a^2 \sin^2\varphi \sin^2\theta}{\sin^2\varphi \cos^2\theta - \cos^2\varphi \sin^2\theta} \, , \tag{2}$$

$$H = \frac{a \sin \varphi \sin \theta}{\sqrt{\sin(\varphi + \theta)\sin(\varphi - \theta)}} \, .$$

[*W. O. Whitescarver.*]

Solved also by H. Y. Benedict, P. H. Philbrick, G. B. M. Zerr, and Jesse Pawling, Jr.

377

PROVE that the rectangle under the focal distances of the origin in the conic represented by the equation $ax^2 + 2hxy + by^2 = 2y$, is $R = 1/(ab - h^2)$.
[*F. P. Matz.*]

SOLUTION.

The foci are given by

$$(Cx + h)^2 = \tfrac{1}{2} \Delta (R + a - b), \quad (Cy - a)^2 = \tfrac{1}{2} \Delta (R + b - a),$$

(See Salmon's Conic Sections, p. 239, exercise and Art. 151) where $C = ab - h^2$, $R^2 = 4h^2 + (a - b)^2$, and Δ, for this exercise, $= - a$.

$$\therefore\ (Cx + h)^2 = \tfrac{1}{2} a (b - a - R) = A^2,$$

$$(Cy - a)^2 = \tfrac{1}{2} a (a - b - R) = B^2;$$

$$\therefore\ x = \frac{-h \pm A}{C}, \quad y = \frac{a \mp B}{C};$$

$$\therefore\ R = \frac{1}{C^2} \sqrt{\{(A + h)^2 + (a + B)^2\}\{(A - h)^2 + (a - B)^2\}}$$

$$= \frac{1}{C} = \frac{1}{ab - h^2}.$$

[*G. B. M. Zerr.*]

Also solved by William Hoover and W. O. Whitescarver.

EXERCISES.

379

FIND the radius of a circle circumscribing the three tangent-circles of radii a, b, and c, respectively. [*F. P. Matz.*]

380

THE sides of a variable rectangle pass through four fixed points. Find the position of the rectangle and its dimensions when its area is a maximum.
[*Geo. R. Dean.*]

381

FROM a point in the circumference of a circle of radius R as centre is described the external arc of a circle of radius r. Determine r so that the area of the lune shall equal that of the original circle. [*W. M. Thornton.*]

382

Four equal circles tangent to each other cut off equal areas from a given circle. Required the radii of the cutting circles when the aggregate area cut off from the given circle is the greatest possible. [*Artemas Martin.*]

383

If c', c'', c''' be the sides of any triangle inscribed in an ellipse, and b', b'', b''' the semi-diameters parallel to the sides, show that the area is

$$A = abc'c''c'''/(4b'b''b''') . \quad [\textit{W. O. Whitescarver.}]$$

384

If c be a chord of an ellipse through the points whose eccentric angles are α and β, and b' the semi-diameter parallel to the chord, show that the area of the triangle formed by the chord and the tangents at its extremities is

$$S = \frac{abc^2}{4b'^2} \tan \tfrac{1}{2} (\alpha - \beta) . \quad [\textit{W. O. Whitescarver.}]$$

385

If in exercise 384 a tangent be drawn parallel to the chord, show that the base of the triangle formed will be $c \sec \frac{1}{2} (\alpha - \beta)$, and its area $ab \tan^3 \frac{1}{2} (\alpha - \beta)$. [*W. O. Whitescarver.*]

386

Prove that the curvature of a bicircular quartic at a point x is the arithmetic mean of the curvature of the four circles which touch the curve at x and pass through the respective real foci. [*F. Morley.*]

387

It is desirable to have a linkwork or other mechanism for keeping four points in the shape of a variable harmonic tetrad, i. e. at the ends of harmonic chords of an arbitrary circle. [*F. Morley.*]

388

An observer, whose eye is at a distance e above the surface of a pond, notices that the setting sun touches a hill-top, distant t from the observer, just as the last of its watery image disappears behind the hill-top reflected in the pond. How high is the hill above the surface of the pond? Give solutions adapted to two cases, (1) when heights are small in comparison with horizontal distances, and (2) when they are not. [*R. A. Harris.*]

389

A portion of xyz space is transformed by means of a rational integral function, algebraic or transcendental; show that ordinarily the XY, XZ, YZ projections of transformed angles which lay originally in planes parallel to

xy, xz, yz, respectively, measure the same as did the original angles—provided the measurement of angles in, or parallel to, YZ or yz be made with an hyperbolic protractor. [*R. A. Harris.*]

390

A VIBRATION is made up of two simple harmonic components, taken in the same direction, whose periods are as 1 : 2, and whose amplitudes are such that the number of maxima and minima of the resultant is the same as of the component of shorter period ; show that the mean amplitude of the resultant is nearly independent of what phase of the one component falls upon a given phase of the other. [*R. A. Harris.*]

391

Two simple harmonic components, taken in the same direction, are combined into one vibration. Show how to determine, by graphic means, the position in time (or angle) of any resultant maximum with reference to the neighboring maximum of the component of shorter period—the relative lengths of the periods being, for successive cases, 1 : 1, 1 : 2, and, if possible, 1 : 3 or more. [*R. A. Harris.*]

392

A SYSTEM of great circles intersects upon the equator of a sphere ; a curve is drawn connecting points on the spherical surface where the circles of this system make a constant angle α with the meridians. Show that the stereographic projection of any such curve is a circular cubic whose equation may be written

$$\tan\alpha\tan\theta = \frac{c^2 + r^2}{c^2 - r^2},$$

c being the radius of the sphere, and one of the poles being the centre of the projection. [*R. A. Harris.*]

393

SHOW that if y be a quadratic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula, $\lambda y_1 + (1 - \lambda) y_2$, where y_1, y_2 correspond to the values

$$x_1 = \frac{h}{2} - h\sqrt{\frac{1}{12}\frac{1-\lambda}{\lambda}}, \quad x_2 = \frac{h}{2} + h\sqrt{\frac{1}{12}\frac{\lambda}{1-\lambda}}.$$

[*Wm. M. Thornton.*]

394

A HETEROGENEOUS rod is hung from a fixed point by two elastic threads of given length fastened at its extremities. Find the position of equilibrium. [*W. H. Echols.*]

395

FIND the difference between the specific heats of a gas whose equation is

$$pv = R\tau - a/\tau v.$$

[*F. P. Matz.*]

CONTENTS.

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